Abstract.—Virtual population and cohort analyses are sometimes conducted with an age category known as the "plus group." This category is used to keep track of the abundance and catches of older fish that cannot be assigned individual ages accurately. In this study we present a procedure for solving the catch equation backwards in time when it involves a plus group. The procedure consists of an initial analytical approximation, followed by a correction function based on an empirical analysis, followed by the application of Newton's Method. The results indicate that the procedure works well for a wide range of natural and fishing mortality values. For comparison, we also applied the same procedure to the basic catch equation (without a plus group) and found that adequate approximation to the true fishing mortality value is achieved even before application of Newton's Method.

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Approximations for solving the catch equation when it involves a "plus group"*

Victor R. Restrepo
Christopher M. Legault
Division of Marine Biology and Fisheries
Rosenstiel School of Marine and Atmospheric Science
University of Miami, 4600 Rickenbacker Causeway, Miami, FL 33149

A solution to $F_{a,t}$ in the catch equation

$$N_{a+1,t+1} = \frac{C_{a,t} Z_{a,t}}{F_{a,t}\,(e^{Z_{a,t}} - 1)}; \qquad (1)$$

where a denotes the age, t denotes the time period, $C_{a,t}$ is the catch in numbers, $N_{a+1,\,t+1}$ is the cohort size at the start of the following time period, $F_{a,t}$ is the fishing mortality rate and $Z_{a,t}$ is the total (fishing + natural) mortality rate ($Z_{a,t} = F_{a,t} + M_{a,t}$), is required in many stock assessment problems that track the exploitation history of a cohort (cohort and virtual population analyses). Here, Equation 1 is expressed in a form known as "backward in time," meaning that a solution to F for age a during time period t is to be obtained given known values of $C_{a,t}$, $N_{a+1,t+1}$, and $M_{a,t}$.

An analytical solution to $F_{a,t}$ is not possible and this prompted several authors to develop useful approximations (see Pope, 1972; Sims, 1982; MacCall, 1986; Allen and Hearn, 1989). Most current computer implementations of age-structured models use numerical algorithms to solve the catch equation; therefore approximations are seldom used in stock assessment applications. However, approximations provide initial estimates that may improve the efficiency of the numerical algorithms, an important consideration when the algorithm is used repeatedly. More important, as with analytical solutions, simple approximations enable scientists to explore relationships between variables easily.

The objective of this study is to present a simple procedure for solving a catch equation that involves a "plus group." A plus group lumps together a number of the oldest age classes in a population into a single age category. For example, a plus group may be used when large (old) fish in the catches cannot be aged with a desired degree of accuracy or precision as can smaller fish (Restrepo and Powers, 1991). We are not aware of definitive analyses that have been conducted to evaluate the general merits of data aggregation into a plus group. Using simulated data with ageng errors in an age-structured model, Fournier and Archibald (1982) found that using a plus group gave better results than did ignoring the ageing errors and extending the analyses to a last "true" age. Deriso et al. (1989), also using simulated data in an age-structured model, found that aggregation of older fish into a plus group produced comparable estimates to those obtained from disaggregated data. However, Hiramatsu (1992) warned that use of a plus group in virtual population analyses could cause large biases under certain cir-

* Contribution No. 94/0501 of the Cooperative Unit of Fisheries Education and Research, Rosenstiel School of Marine and Atmospheric Science, University of Miami, Miami, FL.

cumstances. It seems that the decision of whether or not a plus group should be used in an assessment depends on the ageing errors and characteristics of the population being assessed (Restrepo and Powers, 1991).

The basic plus group dynamics can be presented as follows. Assume that M is the same for the plus group and the previous age (we will drop age and year subscripts for M) and let p denote the plus group age category. The number of fish alive at the beginning of time period t+1 is given by

$$N_{p,t+1} = N_{p,t}\, e^{-Z_{p,t}} + N_{p-1,t}\, e^{-Z_{p-1,t}}, \qquad (2)$$

and the corresponding catch equation is

$$N_{p,t+1} = \frac{C_{p,t}\, Z_{p,t}}{F_{p,t}\,(e^{Z_{p,t}} - 1)} + \frac{C_{p-1,t}\, Z_{p-1,t}}{F_{p-1,t}\,(e^{Z_{p-1,t}} - 1)}. \qquad (3)$$

Given known values of $C_{p,t}$, $C_{p-1,t}$, $N_{p,t+1}$, and M, no unique joint solution exists for the two unknowns, $F_{p,t}$ and $F_{p-1,t}$ in Equation 3. In this study, we use the simplifying assumption that the value of α in

$$\alpha = \frac{F_{p,t}}{F_{p-1,t}} \qquad (4)$$

is known so that $F_{p,t}$ in Equation 3 can be replaced by $\alpha F_{p-1,t}$. Now the solution consists of a single fishing mortality rate, $F_{p-1,t}$. Theoretically, one or more α values can be estimated as parameters in an age-structured model. However, estimation of several α values is difficult, particularly for the last years for which catch data are available (Powers and Restrepo, 1992). Thus, in many assessment applications, α values are assumed from a knowledge of the population and the fishery being examined (Powers and Restrepo, 1992). For example, selectivity studies of the fishing gear may indicate that fish of ages p–1 and older are equally vulnerable, giving α = 1.

Approach

The approach we follow is similar to that used by Sims (1982). We first provide simple approximations to $F_{p-1,t}$, similar to those developed by Pope (1972). On the basis of simulated parameter values, we then empirically estimate correction factors that can be used to improve upon the initial approximations. Finally, we use the corrected approximations as starting guesses for Newton's Method (see Sims, 1982), which can be used to obtain a more accurate numerical solution to the catch equation. For comparative purposes, we also use the same approach for catch equations not involving a plus group.

Range of values examined

In order to evaluate the accuracy of the approximations and to develop empirical corrections, we generated 1,000 uniform random values of each parameter in the following ranges: $N_{p,t}$ [500, 1,500], $N_{p-1,t}$ [500, 1,500], M [0.05, 1.0], $F_{p-1,t}$ [0.05, 3.0], α [0.5, 2.0]. These ranges are arbitrary but we felt that they represent realistic extremes: the stock size of the plus group and the preceding age can differ by a factor of 3; the range in M is representative of a wide range in lifespans; the fishing mortality range is extremely wide because our purpose was to find reasonable approximations for a wide range of F's; and the range in α allows the fishing mortalities of the plus group and the previous age to differ by a factor of 2.

Initial approximations

Pope's (1972) approximation to $F_{a,t}$ in Equation 1 can be explained as follows. Consider the equations (from Appendices A and B in Pope, 1972)

$$N_{a,t} = N_{a+1,t+1}\, e^{Z_t} \qquad (5)$$

and

$$N_{a+1,t+1}\, e^{M} = N_{a,t} - C_{a,t}\, \frac{Z_{a,t}\,(1 - e^{-F_{a,t}})}{F_{a,t}\,(1 - e^{-Z_{a,t}})}. \qquad (6)$$

Pope (1972) demonstrated that, over a range of fishing and natural mortality values, the function multiplying $C_{a,t}$ in Equation 6 can be reasonably approximated by $e^{(M/2)}$. Making use of this approximation and substituting Equation 5 into Equation 6 gives

$$N_{a+1,t+1}\, e^{M} = N_{a+1,t+1}\, e^{Z_{a,t}} - C_{a,t}\, e^{M/2},$$

which can be solved for $F_{a,t}$ as

$$F_{a,t} = \ln\left[\frac{C_{a,t}}{N_{a+1,t+1}} e^{-M/2} + 1\right]. \qquad (7)$$

We followed a similar approach for the purpose of obtaining an initial analytical approximation to $F_{p-1,t}$ in Equation 3. Consider the special case when the fishing mortalities of the plus group and the preceding age are the same (i.e. α=1 in Eq. 4, giving $F_{p,t} = F_{p-1,t}$). Then, the equation analogous to Equation 5 is obtained from Equation 2:

$$[N_{p,t} + N_{p-1,t}] = N_{p,t+1} e^{Z_{p-1,t}}, \tag{8}$$

and the equation analogous to Equation 6 is

$$N_{p,t+1} e^{M} = N_{p,t} - C_{p,t} \frac{Z_{p-1,t}(1-e^{-F_{p-1,t}})}{F_{p-1,t}(1-e^{-Z_{p-1,t}})} + N_{p-1,t} - C_{p-1,t} \frac{Z_{p-1,t}(1-e^{-F_{p-1,t}})}{F_{p-1,t}(1-e^{-Z_{p-1,t}})}, \tag{9}$$

which, using the same approximation given by Pope (1972), becomes

$$N_{p,t+1} e^{M} = [N_{p,t} + N_{p-1,t}] - [C_{p,t} + C_{p-1,t}] e^{M/2}.$$

Substituting Equation 8 into the last equation results in

$$N_{p,t+1} e^{M} = N_{p,t+1} e^{Z_{p-1,t}} - [C_{p,t} + C_{p-1,t}] e^{-M/2},$$

which can be solved for the fishing mortality rate as

$$F_{p-1,t} = \ln\left[\frac{(C_{p,t} + C_{p-1,t})}{N_{p,t+1}} e^{-M/2} + 1\right]. \tag{10}$$

We found that Equation 10 is often a poor approximation for values of $\alpha \neq 1$ (see Results section). A much better approximation can be obtained by introducing α into the equation as

$$F_{p-1,t} = \ln\left[\frac{(C_{p,t}/\alpha + C_{p-1,t})}{N_{p,t+1}} e^{-M/2} + 1\right]. \tag{11}$$

Empirical correction factors

In many cases, given the widespread availability of computers, the approximation given by Equation 11 can be used as an adequate starting guess for an iterative procedure to get a more accurate solution (e.g. see the next section). In some cases, however, it is desirable to improve upon this approximation in order to obtain either a better starting guess, or to obtain as close as possible to an accurate solution because iterative computations are expensive. The latter is the case of solving multiple catch equations while conducting a cohort analysis on a computer spreadsheet and is the motivation for this study.

The empirical approach we used was simple. Denote the approximation in Equation 11 by ${}^{A}F$ and the true fishing mortality by ${}^{T}F$. Our approach was 1) to generate a large number of plausible combinations of parameter values; 2) to plot values of the ratio ${}^{A}F / {}^{T}F$ against α and M (parameters usually assumed to be known a priori) and to identify types of functions that can adequately describe the observed relationship, if any; and 3) to estimate the coefficients of such functions and use them to correct ${}^{A}F$ so it becomes closer to ${}^{T}F$. This empirical approach is similar to the multiple regression approximations suggested by Allen and Hearn (1989). For comparative purposes, we also followed the same approach using the approximation given by Equation 7 for a case without a plus group.

Newton's Method

Sims (1982) suggested the use of Newton's Method for solving F in the catch equation with a desired degree of precision. For Equation 1, the functional equation of interest is

$$f(F_{a,t}) = \frac{C_{a,t} Z_{a,t}}{F_{a,t}(e^{Z_{a,t}} - 1)} - N_{a+1,t+1},$$

and a solution to $F_{a,t}$ is obtained when $f(F_{a,t}) = 0$. Sims (1982) showed that all requirements for convergence were met in order for Newton's Method to converge to that solution. One iteration of Newton's Method (denoted by i) changes the estimate of $F_{a,t}$ as follows:

$$F_{a,t}(i+1) = F_{a,t}(i) - \frac{f\big(F_{a,t}(i)\big)}{f'\big(F_{a,t}(i)\big)} \tag{12}$$

with

$$f'(F_{a,t}) = -\frac{C_{a,t}\left(e^{Z_{a,t}}(F_{a,t}^2 + F_{a,t} M + M) - M\right)}{F_{a,t}^2 (e^{Z_{a,t}} - 1)^2}.$$

For the application of Newton's Method to the solution to the catch equation involving a plus group, the functional equation of interest is (from Equation 3)

$$f(F_{p-1,t}) = \frac{C_{p,t}(\alpha F_{p-1,t} + M)}{\alpha F_{p-1,t}(e^{\alpha F_{p-1,t} + M} - 1)} + \frac{C_{p-1,t}(F_{p-1,t} + M)}{F_{p-1,t}(e^{F_{p-1,t} + M} - 1)} - N_{p,t+1}$$

and its derivative with respect to $F_{p-1,t}$ is

$$f'(F_{p-1,t}) =$$

$$-\frac{C_{p,t}\left(e^{\alpha F_{p-1,t}+M}(\alpha^2 F_{p-1,t}^2 + \alpha F_{p-1,t}M + M) - M\right)}{\alpha F_{p-1,t}^2 (e^{\alpha F_{p-1,t}+M} - 1)^2}$$

$$-\frac{C_{p-1,t}\left(e^{F_{p-1,t}+M}(F_{p-1,t}^2 + F_{p-1,t}M + M) - M\right)}{F_{p-1,t}^2 (e^{F_{p-1,t}+M} - 1)^2}.$$

One iteration of Newton's Method proceeds in the same manner as explained above in Equation 12, if the empirically corrected estimates of fishing mortality (see previous section) as initial values are used. Although we did not carry out a rigorous analysis of conditions for convergence, as Sims (1982) did, we did not encounter any cases where an iteration did not result in an improvement.

Results

Table 1 provides some statistics of the ratio ${}^{A}F/{}^{T}F$ for the 1,000 random combinations of inputs, with and without a plus group. In each case, the first column provides these statistics for the initial approximation (${}^{A}F$ from Equation 7 for the case without a plus group and ${}^{A}F$ from Equations 10 and 11 for the plus group approximation). The next column provides the statistics for the ratios after an empirical correction function is applied (the coefficients of these correction factors are presented in the following subsections). The last two columns give the statistics after one and two iterations of Newton's Method. Implications of these results are explained in more detail below.

Case I: without a plus group

The initial approximation provided by Equation 7 (from Pope, 1972) was reasonable, as expected (See Table 1 and Fig. 1, top panel). The mean ${}^{A}F/{}^{T}F$ ratio indicated an overall 3% bias and the largest error was slightly greater than 8%.

Visual inspection of a plot of ${}^{A}F/{}^{T}F$ against M indicated that a linear relationship would improve the approximation. We fitted the model ${}^{A}F/{}^{T}F = a + bM$ by minimizing the sum of absolute deviations between the observed ratios and those predicted by the model. The parameter estimates were

$$a = 0.9970, \text{ and } b = 0.0808.$$

Thus the empirical correction to the initial approximation to F was

$${}^{emp.A}F = {}^{init.A}F/(a + bM). \qquad (13)$$

Much improvement in the approximation was obtained by use of this simple correction function (Table 1, second column, and Fig. 1, middle panel). The largest observed error was now 3%, which compares favorably with the errors reported by Pope (1972) over a much narrower range of fishing and natural mortality rates. Application of a single iteration of Newton's Method resulted in virtual conver-

Table 1

Summary statistics for 1,000 ratios of the approximated fishing mortality rate to the true fishing mortality rate (${}^{A}F/{}^{T}F$, see text for a description of how the 1,000 realizations were made). Case I represents analyses not using a plus group. Case II represents analyses with a plus group. Each column corresponds to a different step in the analyses, progressing from an initial approximation (two initial approximations are given for Case II), to an empirically corrected approximation, to the first two iterations of Newton's Method. CV = coefficient of variation.

	Case I: no plus group			
	Initial approx. (Eq. 7)	Empirical correction (Eq. 13)	Newton iter. 1 (Eq. 12)	Newton iter. 2 (Eq. 12)
Mean	1.0350	0.9969	1.0000	1.0000
Median	1.0318	1.0000	1.0000	1.0000
CV	0.0217	0.0069	0.0000	0.0000
Min.	1.0007	0.9700	1.0000	1.0000
Max.	1.0826	1.0046	1.0001	1.0000

	Case II: plus group				
	Initial approx. (Eqs. 10, 11)		Empirical correction (Eq. 14)	Newton iter. 1 (Eq. 12)	Newton iter. 2 (Eq. 12)
Mean	1.1193	1.0663	0.9976	0.9983	1.0000
Median	1.1418	1.0660	1.0000	0.9995	1.0000
CV	0.1805	0.0876	0.0368	0.0028	0.0000
Min.	0.6315	0.7654	0.8778	0.9814	0.9994
Max.	1.6757	1.3493	1.1204	1.0000	1.0000

gence to the true F values (Table 1 and Fig. 1, bottom panel).

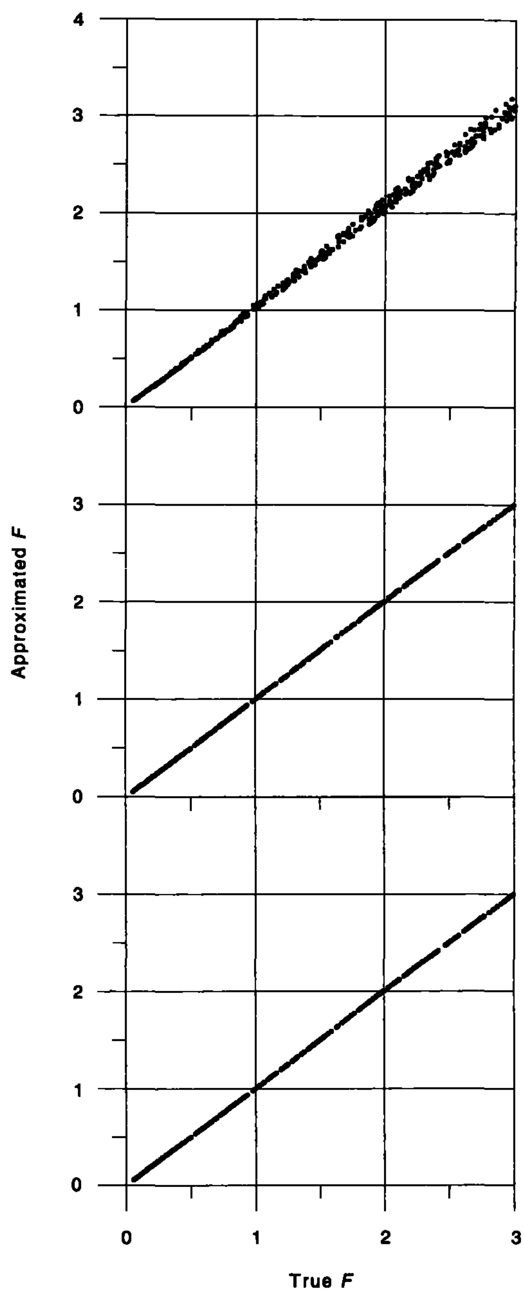

Figure 1

Progression in the approximation to the fishing mortality rate for a catch equation not involving a plus group (400 of the 1,000 pairs of approximated and true F are shown). (**Top**) Initial approximation (Eq. 7); (**Middle**) approximation after application of empirical correction function (Eq. 13); (**Bottom**) approximation after one iteration of Newton's Method (Eq. 12).

Case II: with a plus group

The initial approximations obtained for the plus group problem were rather poor compared with those of the catch equation without a plus group (Table 1, Fig. 2, top panel). This was not unexpected, because the plus group catch equation is not amenable to algebraic manipulations that lead to analytical approximations. However, note that the initial approximation from Equation 11 was much better than that from Equation 10: the observed ${}^{A}F/{}^{T}F$ ratios indicated smaller biases and a tighter approximation overall (see Table 1). (Note: Subsequent statistics and data reported in Table 1 and Figure 2 are based on the approximation given by Equation 11.)

In order to find empirical correction factors, we plotted the observed ${}^{A}F/{}^{T}F$ ratios against M (for different α values) and against α (for different M values) (see Fig. 3). Visual inspection of these figures indicated that 1) the relationship between ${}^{A}F/{}^{T}F$ and M could be approximated by a linear model; 2) the relationship between ${}^{A}F/{}^{T}F$ and α could be approximated by a logarithmic model; and 3) there was an interaction between M and α in terms of explaining variability in ${}^{A}F/{}^{T}F$. Therefore, we fitted the model

$$^{A}F/^{T}F = a_0 + b_1 \ln(\alpha) + b_2 M + b_3 \alpha M$$

to the observed ratios, again by minimizing the sum of absolute residuals. The empirical correction factor used was then

$$^{emp.A}F = {}^{init.A}F/(a_0 + b_1 \ln(\alpha) + b_2 M + b_3 \alpha M) \qquad (14)$$

with

$a_0 = 0.9951$,
$b_1 = 0.2053$,
$b_2 = 0.0636$, and
$b_3 = 0.0161$.

This empirical correction function provided a substantial improvement in the approximations (Table 1, Fig. 2, middle panel). However, solution errors on the order of 12% were still obtained after the correction. One iteration of Newton's Method was sufficient to reduce the errors to within 2% (Table 1 and Fig. 2, bottom panel) and the second iteration resulted in virtual convergence (Table 1).

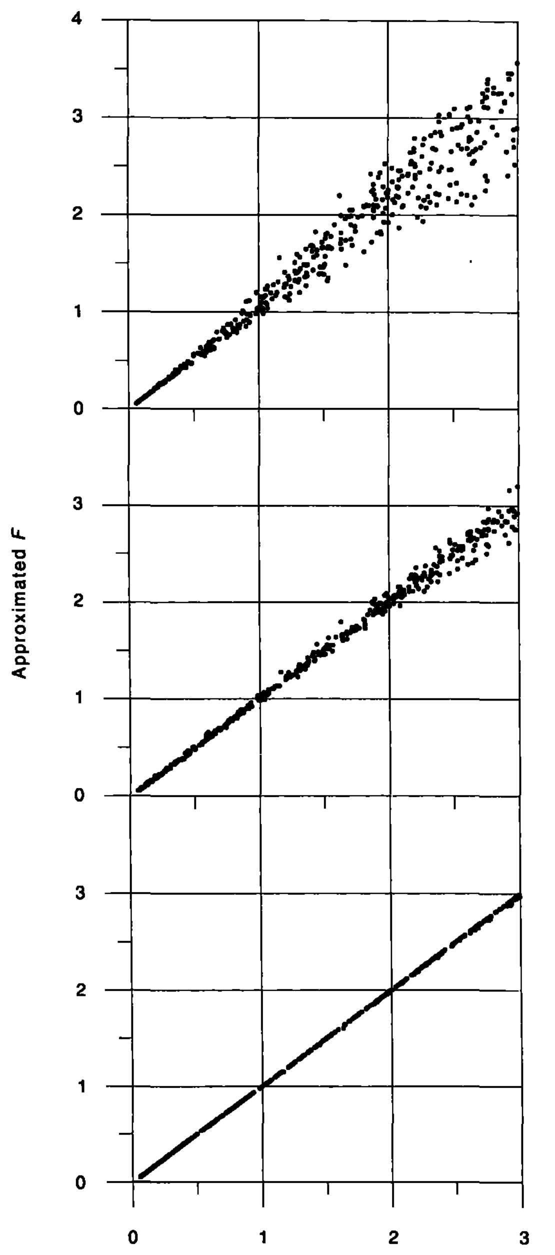

Figure 2

Progression in the approximation to the fishing mortality rate for a catch equation involving a plus group (400 of the 1,000 pairs of approximated and true F are shown). (**Top**) Initial approximation (Eq. 11); (**Middle**) approximation after application of empirical correction function (Eq. 14); (**Bottom**) approximation after one iteration of Newton's Method (Eq. 12).

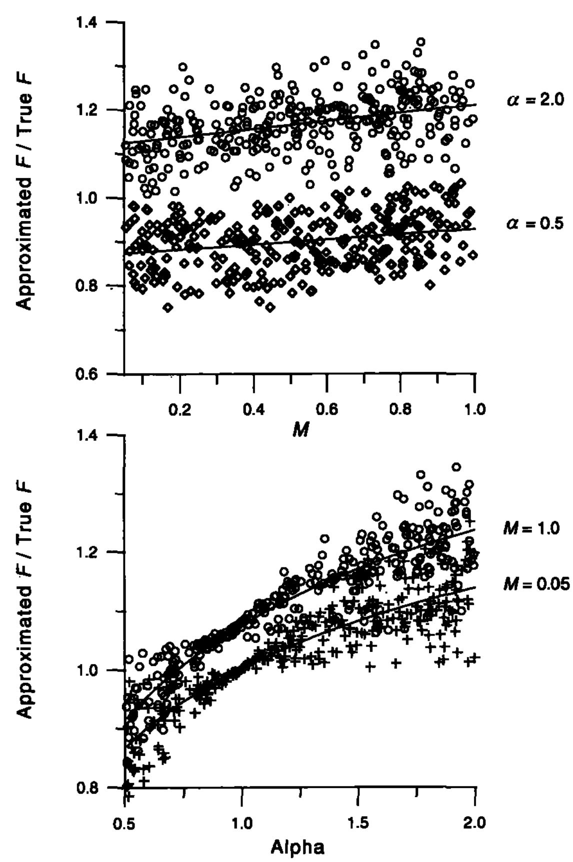

Figure 3

Example of visual analysis to determine the shape of reasonable empirical correction functions. (**Top**) Values of the ratio of initial approximated F to the true F ($^{A}F/^{T}F$) as a function of M for two levels of α, with a linear fit; (**Bottom**) $^{A}F/^{T}F$ values as a function of α for two levels of M, with a logarithmic fit. See text for definition of parameters.

Summary

The results of our study indicate that the empirical correction in Equation 13 applied to Pope's (1972) approximation (Eq. 7) provides an accurate solution to the catch equation that does not involve a plus group (Table 1, Case I). Over a wide range of plausible fishing and natural mortality values, Equation 7 gives errors of up to 8% whereas the empirical correction gives errors of up to 3%. These errors are practically eliminated after one iteration of Newton's Method following the empirical correction.

For the catch equation that involves a plus group, the initial approximations analogous to Pope's (1972) approximation may not be very accurate. For a wide range of plausible mortality values, errors of up to 68% and 35% are obtained from the use of Equations 10 and 11, respectively (Table 1, Case II). The empirical correction in Equation 14 applied to the approximation in Equation 11 reduces the errors to within 12% of the exact solution. One iteration of Newton's Method following the empirical correction reduces the errors to under 2%, and a second iteration practically eliminates the errors.

The computational requirements of our approach for obtaining a solution to the catch equation that includes a plus group are small. If errors up to 12% can be tolerated, the initial approximation (Equation 11) and the subsequent empirical correction (Equation 14) can be incorporated into a single formula. One or two additional computations from the application of Newton's Method substantially reduce or eliminate the biases altogether.

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