

# ON THE RESTRUCTURING OF THE PELLA-TOMLINSON SYSTEM

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## ABSTRACT

The time-dependent analysis of an earlier work is extended to the equilibrium case of the Pella-Tomlinson system, and the relationships between the equilibrium and nonequilibrium versions of the restructured system are developed. The dual formulations of the conventional analysis are avoided and maximum sustainable yield is separated from the indeterminacy of the system. All arbitrary coefficients are eliminated and the management components incorporated directly into the system equations. The source of the statistical degeneracy in the model is revealed and explicitly formulated, and in the companion article by D. Rivard and L. J. Bledsoe (this issue of the *Fishery Bulletin*) the restructured model is treated by a new statistical method that subdues the estimation failures associated with past treatments of the Pella-Tomlinson system.

Because the equilibrium versions of all stock-production models follow from steady-state integrations, the strategy of fishery regulation becomes a strategy of accommodation, so to speak, as determined by a pattern of balanced model states where removals just equal the productivities otherwise surplus to the maintenance needs of the stock. Population status usually enters the process in the simple, robust form of integrated numbers or biomass, and the removals of fishing constitute direct fractions of the whole fishable stock without reference to age or weight distributions. Since the appearance of Schaefer's work (Schaefer 1954) the strategy has been applied to the management of many fisheries. Schaefer devised a rational, linearized method for estimating the parameters of Graham's equilibrium model (Graham 1935) from the actual nonequilibrium yields and effort expenditures of a fishery, a contribution that is often misunderstood. In applying Schaefer's method or like schemes of synthesis, it is not so much that one hopes to observe a pattern of equilibrium levels in a fishery or even expects them to come about, but rather, by knowing the response history of a stock to various exploitation pressures, one might then be guided by the model in bringing a stock, through a sequence of management actions, into a state where some desired level of sustainable yield most likely abides. The philosophy is widely accepted in fisheries management but its application is often censured,

either on economic or biological grounds (see, for example, Larkin 1977).

The exploitation model of Pella and Tomlinson (1969), as it is customarily thought of, extends the more "basic" model of Graham from a system of second degree in nonlinearity to a flexible or more "general" system of indeterminate degree. The increased flexibility comes into the Pella-Tomlinson model through the addition of a single exponential parameter, but the analytical peculiarities that accompany the improvement often lead to paradoxical ends since the equations of the system then permit the simultaneous generation of good data fits and poor parameter estimates (see the commentary of Ricker 1975:323-326 and the treatments of Fox 1971, 1975). This disturbing trait of the statistical model arises from the conflict between the variable (or parametric) curvature of the analytical model and the coupling of that curvature, in the conventional formulations, with all the coefficients of the system. As shown in a prior work (Fletcher 1978), those effects may be separated in the time-dependent analysis by restructuring the system equations so as to accommodate directly the critical-point coordinates of the system graphs. In this paper we extend the analysis to the equilibrium version of the Pella-Tomlinson system, and we show the relationships between the equilibrium model and the (restructured) time-dependent equations.

For a stock of mixed age classes, the most difficult problem in applying any equilibrium model will lie, essentially, in the interpretation of time-dependent transitions between idealized states (however momentary, long-enduring, or

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unobserved such states may be), since the stock will include simultaneously the young and the old, the older having accumulated a probabilistic history of mortality, fecundity, and growth which may differ considerably from the current schedule that affects both. Various tactics for adjusting the parametric mechanics of stock-production models to such long-term, delayed influences are given by Gulland (1969), Fox (1975), Walter,<sup>2</sup> and others, but in the case of the Pella-Tomlinson system the difficulties have been compounded by artifacts of the conventional analysis and by an instability inherent to the mathematical indeterminacy of the system itself. With the critical-point analysis, most of those impediments will convert to tractable relationships or vanish altogether. We can suppress the troublesome dual formulations associated with the conventional casting of the system, we can uncouple the indeterminate exponent and the coefficients of the governing equations, and we can make explicit the relationships between parametric graph curvature and the management components of the system.

### THE REFORMULATED GOVERNING EQUATIONS

Stock-production models, as they are usually defined, arise from the common premise that a fish stock, when reduced by exploitation to a level below some prior abundance, will always strive to recover its former size in accord with some latent, self-regulating mechanism of restoration. Irrespective of the compensatory details, any such recovery must accrue directly from the productivity of the stock, and in the conventional representation of the Pella-Tomlinson system, the latent capacity for biomass production in a stock of fishes is given the dual formulation

$$\dot{P}(B) = \pm aB^n \mp bB, \quad (1a)$$

(1b)

$\dot{P}(B)$  being the production rate of the stock at stock size  $B$ . Equation (1a) applies when exponent  $n$  falls on the range  $0 < n < 1$ , and Equation (1b) applies when  $n > 1$ . In either case, all the critical components of the system—maximum stock size, maximum productivity, the stock level where maximum productivity occurs—depend in some

way on the numerical value assigned to exponent  $n$ . That is, root  $B_\infty$  is given by

$$B_\infty = \left(\frac{a}{b}\right)^{1/1-n},$$

the critical ordinate  $p$  (which corresponds to the stock level where maximum productivity occurs) is determined by

$$p = \left(\frac{an}{b}\right)^{1/1-n},$$

while extremum coordinate  $m$  (which corresponds to productivity  $\dot{P}_{\max}$ ) must be determined from the formula

$$m = \pm \frac{b(1-n)}{n} \left(\frac{an}{b}\right)^{1/1-n},$$

the plus sign applying to Equation (1a) and the minus sign to Equation (1b).

Although exponent  $n$  controls the graph curvatures of Equations (1a) and (1b), the nonzero roots and extrema are controlled by  $B_\infty$  and the coordinate pair  $(p, m)$ . As shown by Fletcher (1975), coordinate  $m$  has no essential dependence on exponent  $n$ , and with the appropriate transformations the dual formulation (Equations (1a, b)) may be suppressed. In consequence, either of the parametric sets  $\{m, p, B_\infty\}$  or  $\{m, n, B_\infty\}$  will constitute a complete set of independent governing parameters for latent productivity in the Pella-Tomlinson system, and the dual formulation (Equations (1a, b)) converts to the single differential equation for latent productivity

$$\dot{P} = \gamma m \left(\frac{B}{B_\infty}\right) - \gamma m \left(\frac{B}{B_\infty}\right)^n, \quad (2)$$

with  $\gamma$  a purely numerical factor wholly prescribed by  $n$  as

$$\gamma = \frac{n^{n/n-1}}{n-1}. \quad (3)$$

With the coefficients so cast, the sign reversals at turning point  $n = 1$  become automatic, and the consolidated interval of definition for  $n$  becomes  $0 < n < \infty$  (the point  $n = 1$  being a removable singularity). With parameter  $m$  thus separated from  $n$  in Equation (2), the undetermined exponent  $n$  can be defined solely by the ratio  $p/B_\infty$  in the relationship

<sup>2</sup>Walter, G. G. 1975. Non-equilibrium regulation of fisheries. Int. Comm. Northwest Atl. Fish. Res. Doc. 75/IX/131, 12 p.

$$\frac{p}{B_\infty} = n^{1/1-n}. \tag{4}$$

When  $n$  takes any value between zero and unity, coordinate  $p$  falls on the range between zero and  $B_\infty/e$ ; when  $n$  takes any value greater than unity, coordinate  $p$  falls on the range between  $B_\infty/e$  and  $B_\infty$ . Wherefore, with  $\{m, p, B_\infty\}$  as the parametric set for Equation (2),  $p$  and  $B_\infty$  determine  $n$ ; with  $\{m, n, B_\infty\}$  as the parametric set,  $n$  and  $B_\infty$  determine  $p$ . In the complete exploitation model, maximum productivity  $m$  becomes maximum sustainable yield (MSY) and biomass level  $p$  becomes the equilibrium level (the " $B_{opt}$ ") where MSY occurs.

For any stock-production system, we may enter exploitation into the productivity formulation by the direct difference  $\dot{P} - \dot{Y}$ , with  $\dot{Y}$  signifying the rate of biomass removal attributed to exploitation. Therefore, in writing

$$\dot{B} = \dot{P} - \dot{Y}, \tag{5}$$

we interpret  $\dot{B}$  as being the resultant productivity that nets to the stock for its growth. When removal rate  $\dot{Y}$  exceeds latent productivity  $\dot{P}$ , then net productivity  $\dot{B} < 0$  and the stock declines; when  $\dot{P}$  exceeds  $\dot{Y}$ , then  $\dot{B} > 0$  and the stock increases. Should  $\dot{Y} = \dot{P}$ , then  $\dot{B} = 0$  and biomass trajectory  $B(t)$  exhibits an extremum, which is the necessary condition for equilibrium fishing. Yield rate  $\dot{Y}$  customarily takes the form

$$\dot{Y}(t) = F(t) \cdot B(t) \tag{6}$$

with the assumption that all fish of the fishable stock share equal probabilities of capture. By admitting Equations (2) and (6) into Equation (5), the differential equation that governs net productivity in the restructured system becomes

$$\dot{B} = \gamma m \left(\frac{B}{B_\infty}\right) - \gamma m \left(\frac{B}{B_\infty}\right)^n - FB \tag{7}$$

and over any time interval, however brief, that mortality  $F$  might be presumed to have a fixed value, biomass variable  $B$  in Equations (6) and (7) has the general time-dependent solution

$$B(t) = \left( B_*^{1-n} + C \exp \left( (\gamma m / B_\infty - F) (1-n) t \right) \right)^{1/(1-n)}, \tag{8}$$

$$B_* = \left( \frac{\gamma m}{\gamma m - F B_\infty} \right)^{1/(1-n)} B_\infty.$$

With initial time  $t_0$  set at zero, the integration constant  $C$  in Equation (8) becomes

$$B_0^{1-n} - B_*^{1-n}.$$

The quantity  $B_*$ , when positive in Equation (8), becomes the adjustment level such that  $B(t) \rightarrow B_*$  over time. When, for certain ranges of  $n$  and  $F$ , quantity  $B_* < 0$ , then the zero root of Equation (7) applies and  $B(t) \rightarrow 0$ . When mortality  $F$  takes the value

$$F_{MSY} \equiv \left( \frac{n-1}{n} \right) \frac{\gamma m}{B_\infty}, \tag{9}$$

irrespective of the value of parameter  $n$ , then  $B(t) \rightarrow p$  and  $\dot{Y} \rightarrow m$  (which are the conditions, in the equilibrium limit, for maximum sustainable yield). In terms of the parameter set  $\{m, p, B_\infty\}$ , Equation (9) becomes, simply,

$$F_{MSY} = \frac{m}{p}.$$

Figure 1 gives a summary of the general constraints on the time-dependent system; for a more detailed treatment of system behavior, see Fletcher (1978).

### THE RESTRUCTURED EQUILIBRIUM SYSTEM

By Equations (2) and (5), the time-varying rate of yield in the reformulated Pella-Tomlinson system takes the form

$$\dot{Y} = \gamma m \left(\frac{B}{B_\infty}\right) - \gamma m \left(\frac{B}{B_\infty}\right)^n - \dot{B}, \tag{10}$$

and when, for given  $F$  and  $n$ , governing Equation (7) exhibits a positive root, then  $B(t) \rightarrow B_*$  and  $\dot{B} \rightarrow 0$  in Equation (10), and yield rate  $\dot{Y}$ , over sufficient time, approaches a constant value. In the steady-state (or "equilibrium") limit, yield then accumulates as

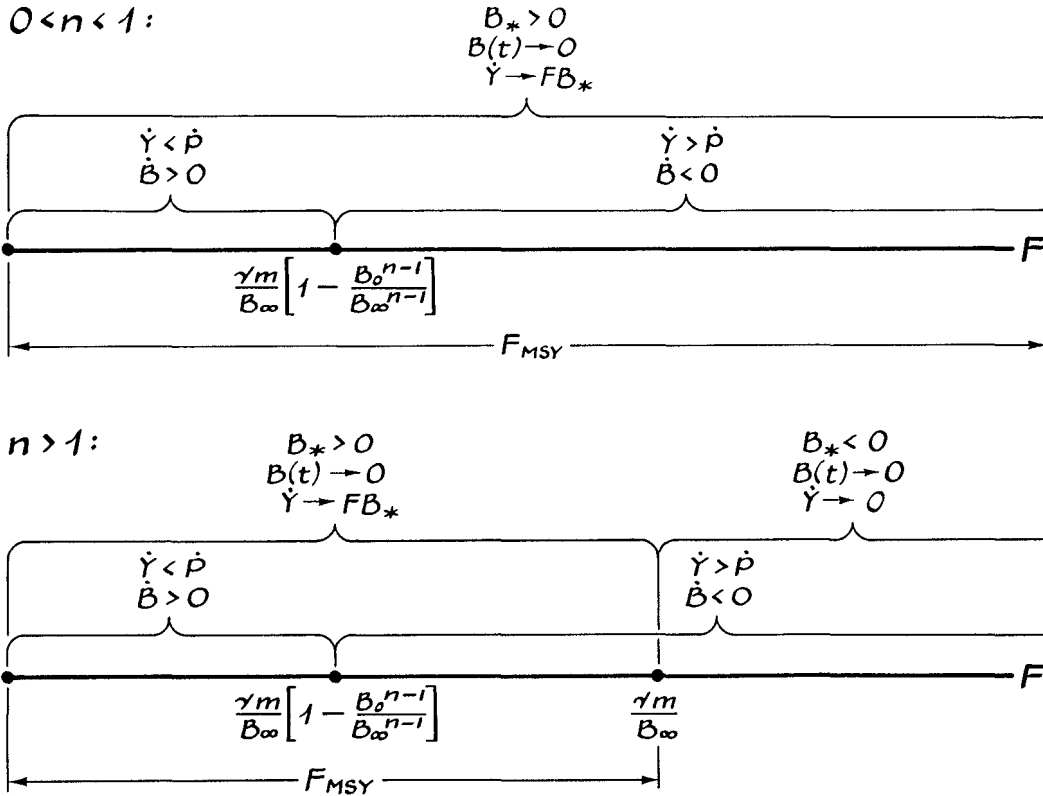


FIGURE 1.—Time-dependent response of the Pella-Tomlinson system to parametric variations of exponent  $n$  and mortality  $F$ . The upper diagram summarizes system response when  $n$  falls on the range  $0 < n < 1$ . The adjustment level of biomass is never zero for this range of  $n$  however great the value of  $F$ , and mortality  $F_{MSY}$  has no absolute constraints; such a stock cannot be fished to extinction. The lower diagram summarizes system behavior when  $n$  falls on the range  $n > 1$ . Mortality  $F_{MSY}$  is then constrained to the interval indicated by the diagram. When  $F$  exceeds the critical value  $\gamma m/B_\infty$ , then the stock, over sufficient time, trends to extinction.

$$\int dY = \int \left( \gamma m \left( \frac{B_*}{B_\infty} \right) - \gamma m \left( \frac{B_*}{B_\infty} \right)^n \right) dt = 0,$$

and for any such equilibrium interval  $\tau$ , the integrated yield rate ( $Y_*/\tau$ ) takes on the parametric formulation

$$\frac{Y_*}{\tau} = \gamma m \left( \frac{B_*}{B_\infty} \right) - \gamma m \left( \frac{B_*}{B_\infty} \right)^n, \quad (11)$$

with maximum latent productivity  $m$  of the time-dependent system becoming the maximum sustainable yield rate (the MSY) of the equilibrium system. With  $B_*$  as the parametric variable in Equation (11), a zero left endpoint exists for  $Y_*/\tau$  when  $n > 1$  and  $F = \gamma m/B_\infty$ . Should  $F$  exceed the critical value  $\gamma m/B_\infty$  when exponent  $n > 1$ , no equilibrium state exists; such conditions in the

time-dependent system correspond to extinction trends. But when  $n$  has any value on the range  $0 < n < 1$ , no left endpoint of Equation (11) exists; a positive equilibrium level of biomass and a non-zero yield rate may be defined for any value of  $F$ , however great.

The equilibrium biomass level  $p$  where MSY (or  $m$ ) occurs can be regulated in Equation (11) by relationship (4). And once designated in (4), the corresponding value of  $n$  determines the value of coefficient  $\gamma$ , as given by Equation (3). Either of the parametric sets  $\{m, p, B_\infty\}$  or  $\{m, n, B_\infty\}$  (augmented by the auxiliary parameters  $F$  and  $B_*$  or  $F$  and  $B_0$ ) will constitute a complete, independent set of controls for equilibrium yield in the Pella-Tomlinson system. Collectively, the parameters control the behavior of equilibrium model Equation (11) but the influence of any one parameter remains independent of the remaining

two. Figure 2 illustrates the individual effects of set  $\{m, p, B_\infty\}$  on the graph of Equation (11).

In equilibrium model (11), biomass level  $B_*$  varies parametrically as a function of equilibrium fishing mortality  $F$ . In terms of parameters  $m, n, B_\infty$ , the relationship becomes

$$B_* = \left( B_\infty^{n-1} - \frac{B_\infty^n F}{\gamma m} \right)^{\frac{1}{n-1}} \quad (12)$$

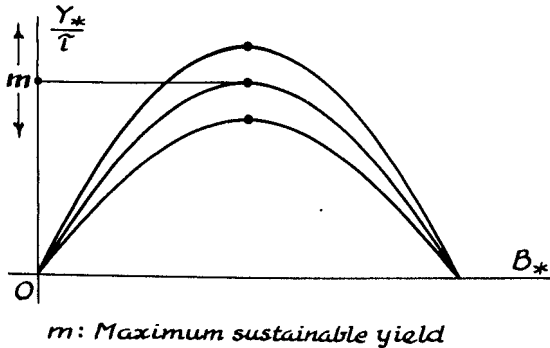
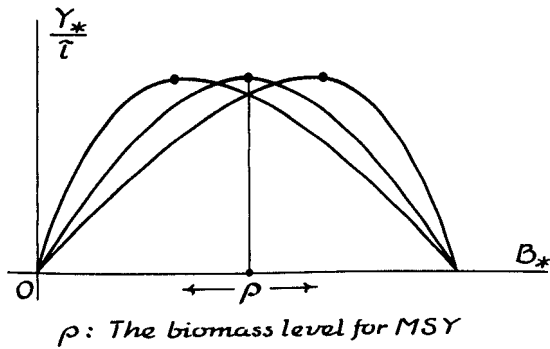
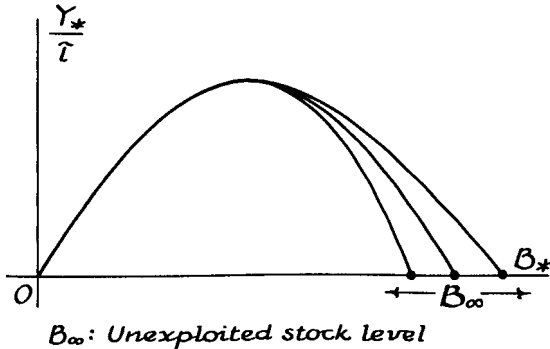


FIGURE 2.—The graph of Equation (11), equilibrium yield vs. equilibrium stock size in the Pella-Tomlinson system, as controlled by independent parameters  $m, p, B_\infty$ .

When exponent  $n$  of the system takes any value on  $0 < n < 1$  then  $\gamma < 0$ , in which case we can see by Equation (12) that  $B_* > 0$  no matter how great the value of  $F$ . That is, when  $0 < n < 1$  there exist equilibrium adjustment levels of stock biomass for all magnitudes of fishing mortality large and small; such a stock defies annihilation. Should exponent  $n > 1$ , however, the corresponding stock can have non-zero adjustment levels  $B_*$  only when  $F < \gamma m / B_\infty$ . That is, when  $n > 1$  and when fishing mortality exceeds the critical value  $\gamma m / B_\infty$ , the "adjustment" level corresponds to extinction and Equation (12) does not apply.

Upon the substitution of Equation (12) into Equation (11), the direct relationship between equilibrium yield and equilibrium fishing mortality becomes

$$\frac{Y_*}{\tau} = \left( F^{n-1} - \frac{B_\infty^n F^n}{\gamma m} \right)^{\frac{1}{n-1}} B_\infty \quad (13)$$

and the fishing mortality that maximizes Equation (13) is given by Equation (9). That is, with the substitution of  $F_{MSY}$  into Equation (13) then  $Y_*/\tau = m$ .

Under the equilibrium conditions, the conventional quantity  $U$  (which signifies accumulated catch per unit of fishing effort as a function of fishing intensity  $f/\tau$ ) can be cast into the restructured form

$$U = \left( U_\infty^{n-1} - \frac{U_\infty^n f}{\gamma m \tau} \right)^{\frac{1}{n-1}} \quad (14)$$

which eliminates the explicit appearance of catchability coefficient  $q$ , permitting instead the direct quantification of maximum sustainable yield  $m$ . Quantities  $U$  and  $U_\infty$  have the customary meanings

$$U \equiv \frac{Y_*}{f} \quad (Y_* \text{ being the yield accumulated over time interval } \tau \text{ as a consequence of effort } f).$$

$$U_\infty \equiv q B_\infty \quad (q \text{ being the individual probability of capture per unit of fishing effort } f).$$

Should the accumulation interval  $\tau$  be a year, the variable  $U$  becomes annual CPUE (catch per unit of effort) and the variable  $f/\tau$  becomes effort per annum. With exponent  $n > 1$  in the Pella-Tomlinson system, no steady-state CPUE exists for a fishing intensity in excess of critical value  $\gamma m / U_\infty$ . But if  $n$

has any value on the interval  $0 < n < 1$ , then steady-state CPUE will exist for all magnitudes of effort (save  $f = \infty$ ). Trajectories of Equation (14) are similar to those of Equation (12), and any graph of (12) will represent the corresponding graph of (14) (CPUE as a function of fishing intensity) with the substitution of  $U$  for  $B_*$ ,  $f/\tau$  for  $F$ , and the ratio  $\gamma m/U_\infty$  for  $\gamma m/B_\infty$ .

## DISCUSSION

For all its adaptability, the Pella-Tomlinson system has serious, inherent limitations and it cannot be viewed as a perfectly generalized model of exploitation-productivity relationships. It is, instead, a nonlinear, first-order, wholly empirical system of open degree that admits of a convenient flexibility in a minimum number of terms. Properly regarded, a particularization of the system will accommodate an arbitrary prototype to the extent that the system graphs might be geometrically accommodating to the data.

Experience with the model has shown that unrealistic estimates of coefficients are likely to occur when the data lie in confined or badly scattered patterns over ranges of effort and yield. The tendency to unrealistic estimates arises from the conflict between graph curvature, as controlled by exponent  $n$ , and the coupling of  $n$  with the coefficients of the system. Heretofore, the exact relationships between exponent  $n$  and the management components have been obscured by the conventional casting of the system. But with the independent parameters and the restructured equations, much of the parametric uncertainty as-

sociated with previous statistical treatments can be circumvented. As we have seen, maximum sustainable yield  $m$  bears no essential relationship to exponent  $n$ , and  $m$  may be wholly separated from  $n$  in all the system equations. And despite the fact that parameters  $m$ ,  $p$ ,  $B_\infty$  share no interdependence (any one may be varied without change in the value of the others), the parametric ratio  $p/B_\infty$  determines  $n$  in the relationship (4). But  $n$  in turn prescribes the curvature (hence the fit) of every graph of the system. As indicated by Figure 3, exponent  $n$  exhibits a dismaying sensitivity to perturbations in ratio  $p/B_\infty$ . The variational response in  $n$ , for a perturbation of  $10^{-1}$  in  $p/B_\infty$ , is of the order of  $n$  near  $n = 1$ , and the instability increases as  $p/B_\infty \rightarrow 1$ . Therefore, when an estimation procedure depends solely on a general curve-fitting statistic, poor parameter estimates are likely to follow, owing to stochastic displacement of datum points at biomass levels remote from locations  $p$  and  $B_\infty$ . In the article that follows, Rivard and Bledsoe (1978) address such problems directly and their work illustrates certain advantages of the restructuring treated here.

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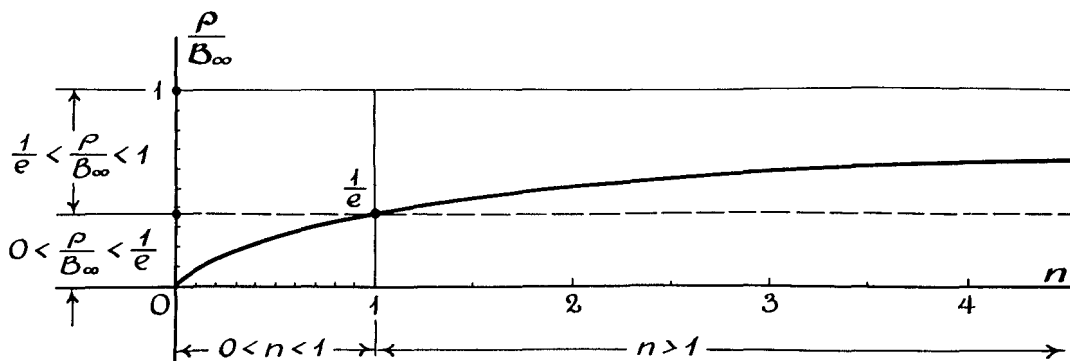


FIGURE 3.—Graph of the relationship between parameter ratio  $p/B_\infty$  and exponent  $n$  of the Pella-Tomlinson system, which indicates the slow convergence of the ratio for increasing  $n$ .

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